## **Lecture 15. Inverse problems of mathematical physics**

We can describe the physical phenomenon by boundary problems for partial differential equations, using physical laws. We have the problem with respect to the state function. However, this problem includes parameters that are supposed known. The direct mathematical physics problem is the problem of finding the state, using known system parameters. However, sometimes parameters of the system are unknown, and we would like to find it, using an information about the state function. This is the inverse mathematical physics problem. It can be solved by means of extremum theory. We consider heat transfer phenomenon as an example.

### **15.1. Direct and inverse problems for the heat transfer phenomenon**

Consider the heat transfer phenomenon. The system is described by the heat equation

  (15.1)

where *u* is the temperature, *t* is the time, *x* is the spatial variable, *c* is the heat capacity, *ρ* is the density, *λ* is the thermal conductivity, and *f* is the density of heat source. The equation (15.1) is considered with initial condition

 *u*(*x*,0) = *u*0(*x*) (15.2)

and the boundary conditions

 *u*(0,*t*) = *p*(*x*), *u*(*L*,*t*) = *q*(*x*), (15.3)

where *u*0 is the initial temperature, *p* and *q* are the boundary temperature, and *L* is the length of the body.

The boundary problem (15.1) – (15.3) has three classes of values. At first, this is the ***state function*** *u.* Secondary, we have the ***independent variables*** *t* and *x.* Finally, we have the ***parameters of the system*** *c*, *ρ*, *λ*, *f*, *u*0, *p*, *q*, *L.* There exists two types of mathematical physics problems. The ***direct mathematical physics problem*** is finding the dependence of the state function of the independent variable with known parameters of the systems. The ***inverse mathematical physics problem*** is finding one or many parameters of the systems on the base of the known information about the state function. The inverse problems are classified by the identifiable and measurable information.

### **15.2. Coefficient inverse problems for the heat transfer phenomenon**

Suppose the thermal conductivity *λ* is unknown. This coefficient is very difficult for measuring for the practice situation. For finding this parameter, it is necessary to know some additional information about the state function, i.e. the temperature. Suppose we can measure the temperature of the body at the concrete points *x*1, *x*1,…, *xn* during the time from zero to *T*, i.e. we have the following additional conditions

 *u*(*xi*,*t*) = *ui*(*t*), 0<*t*<*T*,  *i =*1,2,…,*n*, (15.4)

where the values *ui*, *i =*1,2,…,*n* are known. We have the following ***coefficient inverse problem***. It is necessary to find the parameter *λ* such that the solution *u* of the boundary problem (15.1) – (15.3) satisfies the conditions (15.4).

This problem can be solved, using the methods of extremum problem theory. Determine the functional



Consider the problem of its minimization. Suppose we find the solution of the inverse problem, i.e. the equalities (15.4) are true exactly. Then the functional *I* has the value zero. However, it cannot be negative, because we have only non-negative summand under the integral. Therefore, the solution of the inverse problem is the solution to the extremum problem. Suppose we found the solution of the minimization problem, and the corresponding value of the functional is zero. It can be true if the equalities are true too. In this situation, the solution of the extremum problem is the solution of the inverse problem. Now suppose we determine the solution of the extremum problem, but the corresponding value of the functional is positive. If the inverse problem is solvable, then the minimal value of the functional will be zero. Therefore, the inverse problem is unsolvable. However, the founded solution of the extremum problem can be chosen as the approximate solution of the inverse problem, because now the equalities (15.4) are true with as soon as small error.

Thus, we can transform the inverse problem to the extremum problem. The last problem can be solved, using the known method of extremum theory. This result is true for the different inverse problems.

### **15.3. Source inverse problems for the heat transfer phenomenon**

Suppose the heat source *f* is unknown. Let us have the possibility to measure the heat flux at the boundaries conditions

  (15.5)

where the functions *α* and *β* are known. For the ***source inverse problem***, it is necessary to find the function *f = f*(*x*,*t*) such that the corresponding solution of the problem (15.1) – (15.3) satisfies the equalities (15.5). We can solve this problem by the previous method.

Determine the functional



We have the problem of finding the function *f* that minimizes the functional, where *u* is the solution of the problem (15.1) – (15.3). The relations between the inverse and extremum problems are same as previous case. Then we can solve the inverse problem, using the solution of the extremum problem.

### **15.4. Time inverse problems for the heat transfer phenomenon**

Suppose the boundary temperature *q* at the right end of the bodyis known. Let us have the possibility to measure the heat flux at its left end, i.e. we have the equality

  (15.6)

where the function *β* is known. For the ***boundary inverse problem*** we would like to find the function *q = q*(*x*,*t*) such that the corresponding solution of the problem (15.1) – (15.3) satisfies the equalities (15.6).

Determine the functional



We have the problem of finding the function *q* that minimizes the functional, where *u* is the solution of the problem (15.1) – (15.3). Then we can solve the inverse problem, using the solution of the considered extremum problem.

### **15.5. Boundary inverse problems for the heat transfer phenomenon**

Suppose the initial temperature *u*0at the right end of the bodyis known. Let us have the possibility to measure the temperature at the concrete time *T*, i.e. we have the equality

  (15.7)

where the function *v* is known. For the ***boundary inverse problem*** we would like to find the function *v = v*(*x*) such that the corresponding solution of the problem (15.1) – (15.3) satisfies the equalities (15.7).

Determine the functional



We have the problem of finding the function *v* that minimizes this functional, where *u* is the solution of the problem (15.1) – (15.3). After solving this extremum problem, we can solve the considered inverse problem.